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Geometric structure on the orbit space of gauge connections.

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Abstract. *We investigate the geometric structure of the configuration space for Yang-Mills Field Theory, namely, the structure of the space* $\mathscr{C}^k$ *of connections of Sobolev class* H^k *divided by the action of the gauge group* $\mathscr{G}^{k+1}$, *i.e., the group of* H^{k+1}*-automorphisms of a principal bundle P. The main key is to distinguish in* $\mathscr{G}^{k+1}$ *a subgroup* $\mathscr{G}_0^{k+1}$, *the so-called pointed group, with free action on* $\mathscr{C}^k$ *and to consider the quotient space* $\mathscr{C}^k/\mathscr{G}_0^{k+1}$ *with an action of the compact group* $\mathscr{G}^{k+1}/\mathscr{G}_0^{k+1}$.

For this action we prove a Slice Theorem and a Density Theorem which give rise to the stratification structure of the orbit space, thus also of the configuration space.

INTRODUCTION

For a long time the physical interactions have been described with the aid of gauge fields, connections on suitable principal bundles. The group of automorphisms of the bundle determines orbits in the space of all connections. Then, physicists say that connections belonging to one orbit are different only by a gauge transformation and describe the same physical situation. Hence it is important to investigate the structure of the set of all orbits.

In the present paper we study an action of the whole group of automorphisms

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(gauge group) $\mathcal{G}$ of an arbitrary compact principal bundle P on the space $\mathcal{C}$ of connections seated on P. The quotient space of this action coincides with the configuration space for Yang-Mills Theory.

See e.g. Gribov [10], Singer [19], Narasimhan and Ramadas [16]. The study of the above action was initiated by Singer [19] in 1978, who announced several interesting results for S^n as the base space used by P. Gribov and Singer's results were later investigated by Narasimhan and Ramadas [16] for the case of a trivial $SU(2)$ - bundle over S^3 and S^4. The proofs for a more general case, namely, for nontrivial G-bundles P over M were presented by Mitter and Viallet [15] in 1981. They considered an action of the gauge group on the space $\mathcal{C}_0 \subset \mathcal{C}$ of irreducible connections or an action of the subgroup $\mathcal{G}_0 \subset \mathcal{G}$, consisting of automorphisms fixing a given point of P, a so-called pointed group, on $\mathcal{C}$. Both these cases are simple from the mathematical point of view, since these actions are free. As a result one can not obtain the «true» configuration space for Yang--Mills Theory which coincides with the space of all connections quotiented by the whole group of gauge transformations. This problem was solved in the paper [14] of Kondracki and Rogulski, where it was proved that the configuration space has, in a natural way, a stratification structure onto smooth Hilbert manifolds.

It is well known that from a technical point of view it is convenient to consider the automorphisms of P in the Sobolev class H^{k+1} acting on the H^k-connections, where $k > \frac{1}{2} \dim M + 1$. In this paper we present an approach to the structure of the configuration space from another side. We take the quotient space $\mathcal{C}^k/\mathcal{G}_0^{k+1}$ and act on it by the compact group $\mathcal{G}^{k+1}/\mathcal{G}_0^{k+1}$. It is seen that the topological space $\mathcal{C}^k/\mathcal{G}^{k+1}$ is the same as $\mathcal{C}^k/\mathcal{G}_0^{k+1}$ quotiented by $\mathcal{G}^{k+1}/\mathcal{G}_0^{k+1}$. Since $\mathcal{G}^{k+1}/\mathcal{G}_0^{k+1}$ is compact, this provides some simplifications.

In Section 1 we introduce the notation and some earlier results. The definition is given of $\mathcal{G}_0^{k+1}$ and the action of $\mathcal{G}^{k+1}/\mathcal{G}_0^{k+1}$ on $\mathcal{C}^k/\mathcal{G}_0^{k+1}$ is proved to be smooth. In Section 2 the existence of a slice for the action $\mathcal{G}^{k+1}/\mathcal{G}_0^{k+1}$ on $\mathcal{C}^k/\mathcal{G}_0^{k+1}$ is established. This leads to a suitable decomposition of the manifold $\mathcal{C}^k/\mathcal{G}_0^{k+1}$ upon the countable family of C^∞ Hilbert submanifolds and it follows that the decomposition of the resulting orbit space gives a countable family of C^∞ Hilbert manifolds. In Section 3 we prove a density theorem appropriate for our purposes using essentially the statement proved in [14]. This leads to the stratification structure of the configuration space which is elaborated in Section 4. The resulting stratification is different from and simplier than the stratification obtained in [14]. It means that several strata of the stratification in [14] can form one stratum in the stratification presented here.

1. BASIC DEFINITIONS AND EARLIER RESULTS

We recall here a few definitions and some essential facts to make our paper more self-contained. A smooth G-principal fibre bundle over M is denoted by (P, π, M, G), where P and G are compact. The symbol $\mathscr{G}^k$ will be used for the gauge group of automorphisms of P of Sobolev class H^k, i.e. diffeomorphisms satisfying the following conditions. Let π be the canonical projection $\pi : P \to M$, and let $\varphi \in \mathscr{G}^k$, then we demand that:

(1.1) $\qquad 1° \qquad \pi \circ \varphi = \pi$

(1.2) $\qquad 2° \qquad \forall g \in G, \quad \forall p \in P \quad \varphi(pg) = \varphi(p) g.$

Frequently we shall refer to [14] to illuminate the statements. For each k, $\mathscr{G}^k$ is a Hilbert Lie group and it is also a closed subgroup and submanifold in the full group of H^k-diffeomorphisms $P \to P$. Let $\mathscr{C}^k$ denote the affine spaces of connections on P of Sobolev class H^k, the $\mathscr{C}^k$ are modelled on a Hilbert space.

In what follows, we shall not give explicitly the definition of a connection on P. To specify, if necessary, we shall recall a connection as a $\mathscr{G}$-valued 1-form on P ($\mathscr{G}$ denotes the Lie algebra of G) or as a distribution of horizontal vectors [11] given on P.

For sufficiently large k $\left(\text{i.e. } k > \frac{1}{2} \dim M + 1\right)$, $\mathscr{G}^{k+1}$ is allowed to act on $\mathscr{C}^k$. If we consider a connection a in the sense of a distribution on P, then one can specify this action as follows:

(1.3) $$\mathscr{G}^{k+1} \times \mathscr{C}^k \ni (\varphi, a) \to \varphi a \equiv \varphi_* a \in \mathscr{C}^k.$$

It is known that this action is smooth and proper. Now, we consider a subgroup of symmetries of a given connection $a \in \mathscr{C}^k$

(1.4) $$S_a \subset \mathscr{G}^{k+1}.$$

We say that $\varphi \in S_a$, if and only if, $\varphi a = a$.

Since the action of the gauge group $\mathscr{G}^{k+1}$ on $\mathscr{C}^k$ is proper it follows that S_a is a compact subgroup and thus it is also a Lie group. Moreover as shown e.g. in [14] we may mention the following important fact. If two symmetries $s_1, s_2 \in S_a$ restricted to a chosen fibre P_x over $x \in M$ are equal, then $s_1 = s_2$ on P. Thus, obviously, every symmetry of a connection is completly determined by its values in one arbitrarily chosen fibre of P. Let us take now a point $x_0 \in M$. Let $\mathscr{G}_0^{k+1}$ be a subgroup of $\mathscr{G}^{k+1}$ given by:

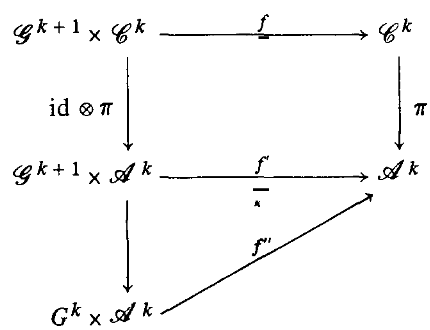
(1.5) $$\mathscr{G}_0^{k+1} = \{\varphi \in \mathscr{G}^{k+1} : \varphi|_{\pi^{-1}(x_0)} = \mathrm{id}\}.$$

We shall call $\mathcal{G}_0^{k+1}$ the pointed group of gauge transformations of the Sobolev class $k+1$. Observe first that there is no symmetry of any connection which is contained in $\mathcal{G}_0^{k+1}$ except for the identity. Moreover $\mathcal{G}_0^{k+1}$ is a normal and closed subgroup of $\mathcal{G}^{k+1}$.

It is easy to see that the quotient group $\mathcal{G}^{k+1}/\mathcal{G}_0^{k+1}$ acts on the fibre $\pi^{-1}(x_0)$ as a group of automorphisms. Choose a point $p_0 \in \pi^{-1}(x_0)$, then for every $\varphi \in \in \mathcal{G}^{k+1}|_{\pi^{-1}(x_0)}$ there exists exactly one $g \in G$ such that $\varphi(p_0) = p_0 g$. Thus, one can distinguish a smooth antiisomorphism between $\mathcal{G}^{k+1}/\mathcal{G}_0^{k+1}$ and the structure group G. Thus $\mathcal{G}^{k+1}/\mathcal{G}_0^{k+1}$ is compact. It is necessary to emphasize that the absence of symmetries of any connection from the subgroup $\mathcal{G}_0^{k+1}$ means that it acts freely on $\mathcal{C}^k$. Since this action is smooth and proper the quotient spaces $\mathcal{C}^k/\mathcal{G}_0^{k+1}$ are separable, paracompact, metrizable Hilbert C^∞--manifolds, see [2]. We can also say that the «essential part» of the action of the gauge group $\mathcal{G}^{k+1}$ on $\mathcal{C}^k$ is involved in the action of quotient group $\mathcal{G}^{k+1}/\mathcal{G}_0^{k+1}$ on $\mathcal{C}^k/\mathcal{G}_0^{k+1}$. For simplicity we introduce the notation:

$$\mathcal{A}^k = \mathcal{C}^k/\mathcal{G}_0^{k+1}, \qquad G^k = \mathcal{G}^{k+1}/\mathcal{G}_0^{k+1}. \tag{1.6}$$

As we have pointed out, the G^k are in 1:1 correspondence to the structure group G. On the other hand, the family of groups G^k does not depend essentially on k. In order to convince ourselves that the last mentioned action is smooth we can argue as follows. Note that thanks to the free action of the pointed group $\mathcal{G}_0^{k+1}$ on $\mathcal{C}^k$, we have a $\mathcal{G}_0^{k+1}$-principal fibre bundle $\mathcal{C}^k$ over $\mathcal{A}^k$. But $\mathcal{G}_0^{k+1}$ is a normal subgroup of $\mathcal{G}^{k+1}$, thus $\mathcal{G}^{k+1}$ acting on $\mathcal{C}^k$ induces an action on the base-space $\mathcal{A}^k$ as well. Now, we consider the diagram:

(1.7) 

where f, f', f'' denote the group-actions on its respective storeys. Since the canonical projection π in $\mathcal{C}^k$ is locally trivial, we can choose a family of smooth local sections s to obtain $\pi \circ f \circ s = f'$. Obviously it does not depend on the selection of s. Finally, f is smooth, thus f' is also, similarly it follows that f'' is smooth.

2. SLICE THEOREM AND ITS CONSEQUENCES

In this section we shall prove the existence of a tubular neighbourhood or equivalently the existence of a slice for the action of G^k on $\mathscr{A}^k$. Then we shall see applications of this. To start the game, however, we first establish the definition.

Let X be a G-space and let Gx denote the orbit of $x \in X$.

DEFINITION. $(U, Pr)_x$ is called a tubular neighbourhood of a given orbit Gx, if and only if, U is a G-invariant, open neighbourhood of Gx and Pr is a locally trivial, G-equivariant retraction of U onto Gx, being a smooth submersion.

By a slice in U we mean $Pr^{-1}(y)$ for $y \in Gx$.

Thus any slice forms a fibre of the bundle (U, Pr, Gx). On the other hand, having one slice only over e.g. $y \in Gx$, we can recover the full tubular neighbourhood of Gx by the action of G. The concept of the slice of G-spaces is one of the most important tools in the analysis of local topological and geometric structures on the orbit space.

THEOREM 2.1. *The action of G^k on $\mathscr{A}^k$ admits a slice for every point $A \in \mathscr{A}^k$.*

The proof which is, in fact, the construction of a tubular neighbourhood in the standard way, needs the following topological lemma.

LEMMA 2.2. *Let X, Y be matric spaces and let a mapping $f: X \to Y$ be a local homeomorphism. Suppose that f is a 1:1 mapping on a closed $B \subset X$ and that $f^{-1} \circ f(B) = B$. Then there exists an open neighbourhood W of B such that*

$$f: W \to f(W) \qquad \text{is a homeomorphism.}$$

For the proof of Lemma, see Bredon [4].

Since $\mathscr{A}^k$ is paracompact, it is possible to define a G^k-invariant, smooth Riemannian metric. In order to do so, take a smooth Riemannian metric γ' and then define

$$\gamma = \int_{G^k} \psi^* \gamma' \, d\mu \tag{2.1}$$

where $\psi \in G^k$, here μ is a Haar measure normalized to one.

Let us choose a point $A \in \mathscr{A}^k$. Let $N \subset T\mathscr{A}^k|_{G^k A}$ be the set of all vectors orthogonal to G^kA. Then N with the natural projection induced from $T\mathscr{A}^k|_{G^kA}$ is a closed subbundle in $T\mathscr{A}^k|_{G^kA}$ because it is the kernel of a smooth and surjective vector bundle morphism, namely the orthogonal projection $T\mathscr{A}^k|_{G^kA} \to TG^kA$. We shall denote the natural G^k-equivariant projection in N by π_N.

Proof. Observe the following identifications:

$$T_A\mathscr{A}^k = \pi_N^{-1}(A) \oplus T_A G^kA = T_0N, \qquad 0 \in \pi_N^{-1}(A), \tag{2.2}$$

where $A \in \mathscr{A}^k$. Since $\exp_*$ is the identity on $T_A\mathscr{A}^k$ and hence also on T_0N, we can find an open neighbourhood $V \subset N$ of zero in the space $\pi_N^{-1}(A)$ such that $\exp : V \to \mathscr{A}^k$ has a derivative being an isomorphism in an arbitrary point $v \in V$. Thus $\exp : V \to \exp(V)$ is a local homeomorphism. Since γ is G^k-invariant, exp is G^k-equivariant and then we can suppose V to be G^k-invariant. Put $X = V$; take B as the image of the zero section in N and $f = \exp$ in Lemma 2.2 then by virtue of it there exists an open G^k-invariant neighbourhood W of the image of the zero section in the bundle N such that $W \subset V$ and $\exp : W \to \exp(W)$ is a homeomorphism, hence also a diffeomorphism. This way we have a tubular neighbourhood $(U, Pr)_A$ of G^kA where $U = \exp W$ and $Pr = \pi_N \exp^{-1}$. ■

When it does not introduce any confusion we shall use the same notation as in (1.4) to denote the isotropy group $S_A \subset G^k$ for a certain point $A \in \mathscr{A}^k$.

Select the following set of vectors in $\pi_N^{-1}(A)$

$$N_A^{S_A} = \{X \in \pi_N^{-1}(A) : \psi \in S_A,\ \psi_*X = X\} \tag{2.3}$$

$N_A^{S_A}$ is a closed vector subspace in $\pi_N^{-1}(A)$.

By the lifting of the action G^k to $T\mathscr{A}^k$ and with the help of this action on $N_A^{S_A}$ we obtain the set $N^{S_A} \subset N$. Obviously, N^{S_A} together with $\pi_N|_{N^{S_A}}$ is a closed smooth subbundle of the bundle N.

Let $\mathscr{A}^k_{(S)}$ be the set of all elements of $\mathscr{A}^k$ with their isotropy groups conjugated to a certain S_A for some fixed point $A \in \mathscr{A}^k$.

THEOREM 2.3. *If $S_A \subset G^k$ is the isotropy group of an element $A \in \mathscr{A}^k$, then $\mathscr{A}^k_{(S)}$ is a smooth submanifold in $\mathscr{A}^k$. Moreover $\mathscr{A}^k_{(S)}$ is G^k-invariant.*

Proof. Since $(U, Pr)_A$ is an open neighbourhood in $\mathscr{A}^k$ of a certain point $A \in \mathscr{A}^k_{(S)}$, it is sufficient to show that $\mathscr{A}^k_{(S)} \cap U$ is a submanifold in U. In fact, we easily have

(2.4) $$\exp(N^{S_A} \cap \exp^{-1}(U)) \subset \mathscr{A}^k_{(S)} \cap U.$$

To show the opposite inclusion, consider the following. Let $A \in \mathscr{A}^k_{(S)} \cap U$ and $\exp^{-1}(A) = X$ thus $X \in N \cap \exp^{-1}(U)$. Then we have for $\psi \in S_A$ $\exp X = A = \psi A = \psi \exp X = \exp \psi_* X$. This means that $X \in N^{S_A}$, since exp is here a G^k-equivariant diffeomorphism. Hence we have the identity

(2.5) $$\mathscr{A}^k_{(S)} \cap U = \exp(N^{S_A} \cap \exp^{-1}(U)).$$

But N^{S_A} is a smooth submanifold in N and exp is a diffeomorphism on $\exp^{-1}(U)$, which together with (2.5) completes the proof. The G^k-invariant property of $\mathscr{A}^k_{(S)}$ is a consequence of N^{S_A} being G^k-invariant. ∎

Remark. The conjugacy classes (S) of closed Lie subgroups $S \subset G^k$ correspond to the submanifolds $\mathscr{A}^k_{(S)}$ in $\mathscr{A}^k$. Observe, in particular, that if S is not an isotropy for any $A \in \mathscr{A}^k$, then $\mathscr{A}^k_{(S)} = \phi$.

LEMMA 2.4. *For any $A \in \mathscr{A}^k$ the bundle N^{S_A} is trivial.*

Proof. We give an explicit form of mapping

(2.6) $$\chi : N^{S_A} \to G^k A \times \pi_N^{-1}(A) \cap N^{S_A}.$$

Let $X \in N^{S_A}$. Choose an element $\psi \in G^k$ in such a way that $\psi \pi_N(X) = A$. Then we put

(2.7) $$\chi(X) \stackrel{\mathrm{df}}{=} (\pi_N(X), \psi_* X).$$

At the same time introduce the notation

(2.8) $$\chi(X) \stackrel{\mathrm{df}}{=} (pr_1(X), pr_2(X))$$

χ does not depend on the choice of ψ, since for two different ψ_1 and ψ_2, there exists $s \in S_{\pi_N(X)}$ such that $\psi_1 = \psi_2 s$. By virtue of the definition of N^{S_A} we have $s_* X = X$. It is easy to verify that χ is a smooth bundle isomorphism. ∎

Remark. Note that for any $\psi \in G^k$ and for any $X \in N^{S_A}$

(2.9) $$pr_2 X = pr_2 \psi_* X;$$

this means that pr_2 is G^k-invariant.

Next consider the orbit space of the action G^k on $\mathscr{A}^k$. We define

(2.10) $$R^k = \mathscr{A}^k / G^k.$$

Then R^k is a topological space which is connected, separable and metrizable. This follows since $\mathscr{A}^k$ is a metrizable space with G^k-invariant Riemannian metric, cf. (2.1).

Let us put

$$R^k_{(S)} = \mathscr{A}^k_{(S)}/G^k. \tag{2.11}$$

Obviously, we have

$$R^k = \bigcup_{S \subset G^k} R^k_{(S)}, \qquad \text{since} \tag{2.12}$$

$$\mathscr{A}^k = \bigcup_{S \subset G^k} \mathscr{A}^k_{(S)} \tag{2.13}$$

where the above sums over all $S \subset G^k$ are disjoint.

Note that for each $A \in \mathscr{A}^k$, there exists anyway the identity as the group of isotropy.

We shall denote by $\hat{\pi}$ the canonical projection $\mathscr{A}^k \to \mathscr{A}^k/G^k$. In the following theorem we show that $R^k_{(S)}$ is endowed, in a natural way, with the C^∞-Hilbert manifold structure.

THEOREM 2.5. *For any $S \subset G^k$ there exists a unique structure of C^∞-Hilbert manifold on $R^k_{(S)}$ such that*

$$\hat{\pi} : \mathscr{A}^k_{(S)} \to R^k_{(S)} \tag{2.14}$$

is a smooth submersion.

Proof. Consider an orbit $G^k A \subset \mathscr{A}^k_{(S)}$ and its tubular neighbourhood $(U, Pr)_A$. Define a chart κ on $V = \hat{\pi}(U \cap \mathscr{A}^k_{(S)})$, namely, for $v \in V$ choose $A' \in \hat{\pi}^{-1}(v)$. Then put

$$\kappa(v) = pr_2 \circ \chi \circ \exp^{-1}(A'). \tag{2.15}$$

This κ does not depend on the choice of A' by Remark (2.9); the mapping $\kappa \circ \hat{\pi} = pr_2 \circ \chi \circ \exp^{-1}$ is continuous thus κ is continuous. Note, moreover that $\kappa^{-1} = \hat{\pi} \circ \exp$. This proves that κ is a homeomorphism. Whenever $V_1 \cap V_2 \subset R^k_{(S)}$ is non-empty, the mapping

$$\kappa_2 \circ \kappa_1^{-1} : W_1 \to W_2 \tag{2.16}$$

where $W_i \subset \pi_N^{-1}(A_i) \cap N^{S_{A_i}}$, $i = 1, 2$, has the form

$$\kappa_2 \circ \kappa_1^{-1} = pr_2 \circ \chi_2 \circ \exp_2^{-1} \circ \exp_1. \tag{2.17}$$

The indices of χ and exp correspond to bundles $N^{S_{A_i}}$, $i = 1, 2$.

It is clear that $\kappa_2 \circ \kappa_1^{-1}$ is smooth, so we have on $R^k_{(S)}$ the structure of a smooth Hilbert manifold. To prove that $\hat{\pi} : \mathscr{A}^k_{(S)} \to R^k_{(S)}$ is a smooth submersion it is sufficient to show that for any chart κ the map $\kappa \circ \hat{\pi}$ is a smooth submersion. But this is clear because

$$\kappa \circ \hat{\pi} = pr_2 \circ (\chi \circ \exp^{-1}) \tag{2.18}$$

is the composition of the diffeomorphism $\chi \circ \exp^{-1}$ and the smooth submersion pr_2. Since $\hat{\pi}$ is now a submersion, the uniqueness of the smooth structure on $R^k_{(S)}$ follows from the general theory of manifolds (see e.g. [2]). ■

This theorem gives the decomposition of the topological space R^k into the disjoint set-theoretical sum of C^∞-Hilbert manifolds $R^k_{(S)}$.

PROPOSITION 2.6. *The decomposition of R^k upon $R^k_{(S)}$ is countable.*

Proof. Respective manifolds $R^k_{(S)}$ are labeled by suitable conjugacy classes of isotropy groups. But in a compact Lie group the number of conjugacy classes of closed subgroups is countable, see [12]. ■

3. DENSITY THEOREM

The aim of this section is to show that if we consider two submanifolds $\mathscr{A}^k_{(S)}$, $\mathscr{A}^k_{(S')} \subset \mathscr{A}^k$ for $S' \subset S$, then $\mathscr{A}^k_{(S')}$ is dense in $\mathscr{A}^k_{(S)} \cup \mathscr{A}^k_{(S')}$. Similarly for the manifolds $R^k_{(S)} \subset R^k$. Let us recall several relevant things.

For a given connection $a \in \mathscr{C}^k$ we denote its holonomy group in a point $x_0 \in M$ by $H^a(x_0)$. As is well known, this is a group of automorphisms of the fibre $\pi^{-1}(x_0)$, and since, as mentioned before in Section 1, G^k is the group of all automorphisms of $\pi^{-1}(x_0)$, we have $H^a(x_0) \subset G^k$.

Choose $p_0 \in \pi^{-1}(x_0)$. Consider the set of all $p \in P$ which can be jointed with p_0 by a horizontal curve with respect to the connection a. If M is simply connected, then the above set is a closed subbundle of the bundle P. This subbundle we shall call the holonomy bundle of the connection $a \in \mathscr{C}^k$ and denote it by $\mathscr{H}^a(p_0)$. See [17] for details. The structure group of this subbundle is a subgroup of G its action on $p_0 \in \pi^{-1}(x_0)$ yields $\mathscr{H}^a(p_0) \cap \pi^{-1}(x_0)$. On the other hand (cf. Section 1) for fixed point $p_0 \in \pi^{-1}(x_0)$ we have an antiisomorphism $G^k \to G$, which allows us to set up a correspondence between the structure group in $\mathscr{H}^a(p_0)$ and the holonomy group $H^a(x_0)$ for chosen $p_0 \in \pi^{-1}(x_0)$.

We have the useful fact:

THEOREM 3.1. *Let M be simply-conected manifold* $\dim M \geqslant 2$ *and let* $k > \frac{1}{2}\dim M + 1$. $\mathscr{H}_1 \subset \mathscr{H}_2$ *are connected subbundles in P. Then every H^k-connection a on P, with the holonomy bundle $\mathscr{H}^a(p_0) = \mathscr{H}_1$ can be approximated, in the sense of the H^k-topology on $\mathscr{C}^k$, by a sequence of H^k-connections a_n on P with their holonomy bundles $\mathscr{H}^{a_n}(p_0) = \mathscr{H}_2$ for some fixed point $p_0 \in P$.*

For the proof of this theorem, see [14].

Let the connection a be contained in the orbit $A \subset \mathscr{C}^k$. Then observe, that

$$S_a|_{\pi^{-1}(x_0)} = S_A. \tag{3.1}$$

In fact, taking two connections $a_1, a_2 \in A$ and $\varphi \in \mathscr{G}_0^{k+1}$ such that $\varphi a_1 = a_2$ we obtain $\varphi S_{a_1} \varphi^{-1} = S_{a_2}$. But $\varphi|_{\pi^{-1}(x_0)} = \mathrm{id}$, thus

$$(\varphi S_{a_1} \varphi^{-1})|_{\pi^{-1}(x_0)} = S_{a_1}|_{\pi^{-1}(x_0)} = S_{a_2}|_{\pi^{-1}(x_0)}.$$

For $G' \subset G$, CG' denotes the centralizer of subgroup G' in G. The following straightforward algebraic lemma will be useful.

LEMMA 3.2. *Let G_1, G_2 be subgroups of G, such that $CG_1 \subset CG_2$. Then $C(G_1G_2) = CG_1$, where by G_1G_2 we mean the subgroup of G spanned by G_1 and G_2.*

LEMMA 3.3. *For any connection $a \in A$, with an orbit $A \in \mathscr{A}^k$,*

$$S_A = CH^a(x_0). \tag{3.2}$$

Proof. Consider $\varphi_{x_0} \in G^k$. Let σ denote a smooth curve in M starting at $x_0 \in M$ and with end point at $x \in M$. By h_σ we shall denote the parallel translation along any curve σ. Hence h_σ is a morphism of the fibre $\pi^{-1}(x_0)$ onto the fibre $\pi^{-1}(x)$. Define an automorphism of the fibre $\pi^{-1}(x)$ by:

$$\varphi_x = h_\sigma \varphi_{x_0} h_\sigma^{-1}. \tag{3.3}$$

Note that if $\varphi_{x_0} \in CH^a(x_0)$, then φ_x is independent of a choice of σ. In fact, if we take two curves σ_1 and σ_2 such that

$$\varphi_x = h_{\sigma_1} \varphi_{x_0} h_{\sigma_1}^{-1} = h_{\sigma_2} \varphi_{x_0} h_{\sigma_2}^{-1}, \tag{3.4}$$

then also

$$\varphi_x = h_{\sigma_2} h_{\sigma_2}^{-1} h_{\sigma_1} \varphi_{x_0} h_{\sigma_1}^{-1} h_{\sigma_2} h_{\sigma_2}^{-1} \tag{3.5}$$

but $h_{\sigma_2}^{-1} h_{\sigma_1}$ is an element of the holonomy group with reference curve $\sigma_1 \cup \sigma_2^{-1}$. Now, varying an arbitrary point $x \in M$ we obtain an automorphism of P determined by its action in the fibre $\pi^{-1}(x_0)$. One can verify with the aid of local trivialization of P that the automorphism so obtained is a symmetry of the connection a. For details we refer to [14]. In view of (3.1), the formula (3.2) holds. ∎

Now, we proceed to prove the main theorem of this Section.

THEOREM 3.4. *Let* $\dim M \geqslant 2$, *let* $S_A \subset G^k$ *be an isotropy group of* $A \in \mathscr{A}^k$. *Suppose that* $S' \subset S_A$ *with* $A' \in \mathscr{A}^k$ *such that* S' *is its isotropy group. Then there exists sequence* $\{A_n\}$, $A_n \xrightarrow[n\to\infty]{} A$ *in* $\mathscr{A}^k$ *with* $S_{A_n} = S'$.

Proof. Choose the connections a, a' from the orbits A, $A' \in \mathscr{A}^k$ respectively. For $p_0 \in \pi^{-1}(x_0)$ let us take all parallel translations of p_0 with respect to the connections a and a' along all curves starting at x_0 and all their compositions. The set obtained in this way defines some connected subbundle $\hat{\mathscr{H}}$ of P. Note that

$$(3.6) \qquad \mathscr{H}^a(p_0) \subset \hat{\mathscr{H}} \subset P.$$

By virtue of Theorem 3.1, there exists $a_n \to a$ in $\mathscr{C}^k$ such that $\mathscr{H}^{a_n}(p_0) = \hat{\mathscr{H}}$. By the construction of $\hat{\mathscr{H}}$, the holonomy group corresponding to connections a_n coincides with $H^a(x_0)H^{a'}(x_0)$. Moreover by continuity of the projection $\pi : \mathscr{C}^k \to \mathscr{A}^k$ we obtain the sequence $A_n \xrightarrow[n\to\infty]{} A$ in $\mathscr{A}^k$. But by lemma 3.2. $C(H^a(x_0)H^{a'}(x_0)) = CH^{a'}(x_0)$ and by lemma 3.3. $CH^{a'}(x_0) = S_{A'} = S'$. Hence we complete the proof with $S_{A_n} = S'$. ∎

Let J denote the set of conjugacy classes of all isotropy groups in G^k. This set can be endowed with the following ordering. For $(S_1), (S_2) \in J$ put

$$(3.7) \qquad (S_1) < (S_2) \Leftrightarrow \text{there are } S_1 \in (S_1) \text{ and } S_2 \in (S_2) \text{ such that } S_1 \subset S_2.$$

From the definition of conjugacy class we have immediately that each group $S_2 \in (S_2)$ contains some group from the class (S_1).

COROLLARY 3.5. *For* $(S_1), (S_2) \in J$

$$(3.8) \qquad (S_1) < (S_2) \Rightarrow \mathscr{A}^k_{(S_1)} \text{ is dense in } \mathscr{A}^k_{(S_1)} \cup \mathscr{A}^k_{(S_2)},$$

$$(3.9) \qquad (S_1) < (S_2) \Rightarrow R^k_{(S_1)} \text{ is dense in } R^k_{(S_1)} \cup R^k_{(S_2)}.$$

Use Theorem 3.4 and (3.7) for the proof of this Corollary.

4. STRATIFICATION STRUCTURE OF $\mathbf{R}^k$

In this section we shall see that the decomposition of the topological space R^k into smooth Hilbert manifolds $R^k_{(S)}$ is a stratification.

By the stratification of a topological space X we mean:

DEFINITION. Let D be a countable (or finite) disjoint family of non-empty subsets of a topological space X. We require

(4.1) $\quad UD = X$ and

for $\Omega, \Omega' \in D$

(4.2) $\quad \overline{\Omega} \cap \Omega' \neq \phi \Rightarrow \Omega' \subset \overline{\Omega}.$

If moreover each $\Omega \in D$ has a smooth Hilbert manifold structure compatible with the topology induced from X then we shall call D a *stratification* of the topological space X into smooth Hilbert manifolds. ■

(4.3) An element $\Omega \in D$ is called a *stratum*.

Every stratification D distinguishes a partial ordering given as follows.

For $\Omega, \Omega' \in D$

(4.4) $\quad \Omega_1 < \Omega_2 \Leftrightarrow \overline{\Omega}_1 \cap \Omega_2 \neq \phi.$

It is easy to see that $<$ is a reflexive and transitive relation. Moreover for each stratification D of X and for a stratum $\Omega \in D$, we have

(4.5) $$\overline{\Omega} = \bigcup_{\Omega < \Omega'} \Omega'$$

THEOREM 4.1. *The family $D = \{R^k_{(S)} : (S) \in J\}$ is a stratification of R^k.*

Proof. It is sufficient to prove the assertion (4.2) since the others are the consequences of (2.12), Theorem 2.5 and Proposition 2.6.

Consider $\mathscr{A}^k_{(S)} = \hat{\pi}^{-1}(R^k_{(S)})$ and choose the set of orbits $\mathscr{A}^k_S \subset \mathscr{A}^k_{(S)}$ such that the isotropy group of $A \in \mathscr{A}^k_S$ is exactly S. Observe, that $\hat{\pi}(\mathscr{A}^k_S) = R^k_{(S)}$. Let us verify that

(4.6) $$\hat{\pi}(\overline{\mathscr{A}^k_S}) = \overline{\hat{\pi}(\mathscr{A}^k_S)}.$$

The inclusion from left to right is obvious. To see the converse inclusion note that

$\hat{\pi}$ is closed, since we deal with the action of compact group G^k. (See [4]). Thus $\hat{\pi}(\overline{\mathscr{A}^k_S})$ is closed and contains $\hat{\pi}(\mathscr{A}^k_S)$, hence it contains also $\overline{\hat{\pi}(\mathscr{A}^k_S)}$.

Now, we prove that $\overline{\mathscr{A}^k_S}$ contains points with the isotropy group $S' \supseteq S$. This is true, because taking the convergent sequence $A_k \in \mathscr{A}^k_S$ and acting on it with an element of S we obtain the isotropy group $S' \supseteq S$ of the limit $A \in \overline{\mathscr{A}^k_S}$. From (4.6) we have

$$\hat{\pi}(\overline{\mathscr{A}^k_S}) = \overline{\hat{\pi}(\mathscr{A}^k_S)} = \overline{R^k_{(S)}}. \tag{4.7}$$

Now, if $R^k_{(S')} \cap \overline{R^k_{(S)}} = \phi$ then $S \subseteq S'$ and also $(S) < (S')$. By the virtue of Corollary 3.5 $R^k_{(S)}$ is dense in $R^k_{(S)} \cup R^k_{(S')}$. Thus $R^k_{(S)} \cup R^k_{(S')} \subset \overline{R^k_{(S)}}$ and the $R^k_{(S')} \subset \overline{R^k_{(S)}}$. ■

Remarks

Analyzing the above proof we easily get that

$$R^k_{(S)} < R^k_{(S')} \Leftrightarrow (S) < (S') \tag{4.8}$$

where we have used a consistent ordering relation for both sides.

In [14] a stronger notion of stratification was investigated, namely, it was demanded that

$$\Omega, \Omega' \in D, \quad \overline{\Omega} \cap \Omega' \neq \phi \Rightarrow \overline{\Omega}' \cap \Omega = \phi. \tag{4.9}$$

A stratification which fulfils this condition is called a regular stratification. (4.9) assures that the partial ordering in the set of strata becomes an ordering. Also it was shown in [13] that every stratification of a Hausdorff space is regular. Of course R^k is a Hausdorff space, since it is matrizable.

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